

LIPSCHITZIAN SOLUTIONS TO INHOMOGENEOUS LINEAR ITERATIVE EQUATIONS

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ABSTRACT. We study the problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ of equations of the form

$$\varphi(x) = \int_{\Omega} g(\omega) \varphi(f(x, \omega)) \mu(d\omega) + F(x),$$

where μ is a measure on a σ -algebra of subsets of Ω .

1. INTRODUCTION

Fix a measure space $(\Omega, \mathcal{A}, \mu)$ and a separable metric space (X, ρ) .

Motivated by appearance of the equation

$$\varphi(x) = \int_{A_1} \varphi(f(x, \omega)) \mu(d\omega) + c - \int_{A_2} \varphi(f(x, \omega)) \mu(d\omega)$$

with disjoint $A_1, A_2 \in \mathcal{A}$ in the theory of perpetuities and of refinement equations, see section 3.4 of the survey paper [3], we consider problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ to the equation

$$(1) \quad \varphi(x) = \int_{\Omega} g(\omega) \varphi(f(x, \omega)) \mu(d\omega) + F(x).$$

Concerning the given functions f, g and F we assume the following hypotheses in which \mathcal{B} stands for the σ -algebra of all Borel subsets of X and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(H₁) Function f maps $X \times \Omega$ into X and for every $x \in X$ the function $f(x, \cdot)$ is \mathcal{A} -measurable, i.e.,

$$\{\omega \in \Omega : f(x, \omega) \in B\} \in \mathcal{A} \quad \text{for all } x \in X \text{ and } B \in \mathcal{B}.$$

(H₂) Function $g : \Omega \rightarrow \mathbb{K}$ is integrable,

$$\int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) < \infty \quad \text{for every } x \in X,$$

and

$$(2) \quad \int_{\Omega} |g(\omega)| \rho(f(x, \omega), f(z, \omega)) \mu(d\omega) \leq \lambda \rho(x, z) \quad \text{for all } x, z \in X$$

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with a $\lambda \in [0, 1)$.

(H₃) Function F maps X into a separable Banach space Y over \mathbb{K} and

$$(3) \quad \|F(x) - F(z)\| \leq L\rho(x, z) \quad \text{for all } x, z \in X$$

with an $L \in [0, +\infty)$.

As emphasized in [4, section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. However, as it seems, Lipschitzian solutions are examined rather by the fixed-point method (cf. [4, section 7.2D]). We iterate the operator which transforms a Lipschitzian $F: X \rightarrow Y$ into $\int_{\Omega} g(\omega)F(f(x, \omega))\mu(d\omega)$; cf. formulas (6) and (8) below. The spacial case where $g(\omega) = -1$ for every $\omega \in \Omega$ and $\mu(\Omega) = 1$ was examined in [2] on a base of iteration of random-valued functions.

Integrating vector functions we use the Bochner integral.

2. EXISTENCE AND UNIQUENESS

Putting

$$(4) \quad \gamma = \int_{\Omega} g(\omega)\mu(d\omega),$$

we start with two simple lemmas.

Lemma 2.1. *Assume (H₁) and let $g: \Omega \rightarrow \mathbb{K}$ be integrable with $\gamma \neq 1$. If (2) holds with a $\lambda \in [0, 1)$, then for any F mapping X into a normed space Y over \mathbb{K} equation (1) has at most one Lipschitzian solution $\varphi: X \rightarrow Y$.*

Proof. Fix a function F mapping X into a normed space Y over \mathbb{K} , let $\varphi_1, \varphi_2: X \rightarrow Y$ be Lipschitzian solutions of (1), and put $\varphi = \varphi_1 - \varphi_2$. Then φ is a Lipschitzian solution of (1) with $F = 0$, and denoting by L_{φ} the smallest Lipschitz constant for φ , by (2) for all $x, z \in X$ we have

$$\|\varphi(x) - \varphi(z)\| \leq \int_{\Omega} |g(\omega)| \|\varphi(f(x, \omega)) - \varphi(f(z, \omega))\| \mu(d\omega) \leq L_{\varphi} \lambda \rho(x, z),$$

whence $L_{\varphi} = 0$ and φ is a constant function. Since γ defined by (4) is different from 1, the only constant solution of (1) with $F = 0$ is the zero function. \square

Lemma 2.2. *Under the assumptions (H₁)–(H₃) for every $x \in X$ the function*

$$\omega \mapsto g(\omega)F(f(x, \omega)), \quad \omega \in \Omega,$$

is Bochner integrable and

$$(5) \quad \left\| \int_{\Omega} g(\omega)F(f(x, \omega))\mu(d\omega) - \int_{\Omega} g(\omega)F(f(z, \omega))\mu(d\omega) \right\| \leq L\lambda\rho(x, z)$$

for all $x, z \in X$.

Proof. The function considered is \mathcal{A} -measurable, for every $\omega \in \Omega$ we have

$$\|g(\omega)F(f(x, \omega))\| \leq L|g(\omega)|\rho(f(x, \omega), x) + L|g(\omega)|\|F(x)\|,$$

and (5) holds for all $x, z \in X$. \square

Assuming (H_1) – (H_3) and applying Lemma 2.2 we define

$$(6) \quad F_0(x) = F(x), \quad F_n(x) = \int_{\Omega} g(\omega)F_{n-1}(f(x, \omega))\mu(d\omega)$$

for all $x \in X$ and $n \in \mathbb{N}$, and we see that

$$(7) \quad \|F_n(x) - F_n(z)\| \leq L\lambda^n \rho(x, z) \quad \text{for all } x, z \in X \text{ and } n \in \mathbb{N}.$$

Our main result reads.

Theorem 2.3. *Assume (H_1) – (H_3) . If $\gamma \neq 1$ then equation (1) has exactly one Lipschitzian solution $\varphi: X \rightarrow Y$; it is given by the formula*

$$(8) \quad \varphi(x) = \frac{1}{1-\gamma} \left(\sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + F(x) \right) \quad \text{for every } x \in X,$$

$$(9) \quad \|\varphi(x) - \varphi(z)\| \leq \frac{L(1+|\gamma|)}{|1-\gamma|(1-\lambda)} \rho(x, z) \quad \text{for all } x, z \in X,$$

and

$$(10) \quad \|\varphi(x)\| \leq \frac{1}{|1-\gamma|} \left(\frac{L}{1-\lambda} \int_{\Omega} |g(\omega)|\rho(f(x, \omega), x)\mu(d\omega) + \|F(x)\| \right)$$

for every $x \in X$.

Proof. For the proof of the existence observe first that by (4), (6) and (7) for all $x \in X$ and $n \in \mathbb{N}$ we have

$$(11) \quad \begin{aligned} \|F_n(x) - \gamma F_{n-1}(x)\| &= \left\| \int_{\Omega} g(\omega)F_{n-1}(f(x, \omega))\mu(d\omega) - \int_{\Omega} g(\omega)F_{n-1}(x)\mu(d\omega) \right\| \\ &\leq L\lambda^{n-1} \int_{\Omega} |g(\omega)|\rho(f(x, \omega), x)\mu(d\omega). \end{aligned}$$

Consequently (8) defines a function $\varphi: X \rightarrow Y$. Routine calculations, (8), (7), (2) and (11) show that this function satisfies (9) and (10).

It remains to prove that φ solves (1). To this end define $M: X \rightarrow [0, \infty)$ by

$$(12) \quad M(x) = L \int_{\Omega} |g(\omega)|\rho(f(x, \omega), x)\mu(d\omega)$$

and fix $x_0 \in X$. An obvious application of (12), (H_2) , (10) and (3) gives

$$(13) \quad M(x) \leq c_1 \rho(x, x_0) + c_2, \quad \|\varphi(x)\| \leq c_1 \rho(x, x_0) + c_2 \quad \text{for every } x \in X$$

with some constants $c_1, c_2 \in [0, \infty)$.

Fix $x \in X$. According to Lemma 2.2 the function

$$\omega \mapsto g(\omega)\varphi(f(x, \omega)), \quad \omega \in \Omega,$$

is Bochner integrable. Moreover, by (11)–(13),

$$\begin{aligned} \left\| g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \right\| &\leq \lambda^{n-1} |g(\omega)| M(f(x, \omega)) \\ &\leq \lambda^{n-1} |g(\omega)| (c_1 \rho(f(x, \omega), x_0) + c_2) \\ &\leq \lambda^{n-1} |g(\omega)| (c_1 \rho(f(x, \omega), x) + c_1 \rho(x, x_0) + c_2) \end{aligned}$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Hence, making use of (H₂), the dominated convergence theorem and (6) we see that

$$\begin{aligned} (14) \quad &\int_{\Omega} \sum_{n=1}^{\infty} g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \mu(d\omega) \\ &= \sum_{n=1}^{\infty} \int_{\Omega} g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \mu(d\omega) \\ &= \sum_{n=1}^{\infty} (F_{n+1}(x) - \gamma F_n(x)). \end{aligned}$$

Applying now (8), (14) and (6) we get

$$\begin{aligned} &\int_{\Omega} g(\omega) \varphi(f(x, \omega)) \mu(d\omega) \\ &= \frac{1}{1-\gamma} \int_{\Omega} \left[\sum_{n=1}^{\infty} g(\omega) \left(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)) \right) \right. \\ &\quad \left. + g(\omega) F(f(x, \omega)) \right] \mu(d\omega) \\ &= \frac{1}{1-\gamma} \left[\sum_{n=1}^{\infty} (F_{n+1}(x) - \gamma F_n(x)) + F_1(x) \right] \\ &= \frac{1}{1-\gamma} \left[\sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + \gamma F(x) \right] \\ &= \varphi(x) - F(x). \end{aligned}$$

The proof is complete. □

3. EXAMPLES

Example 3.1. Given $\lambda \in (0, 1)$ and an integrable $\xi: \Omega \rightarrow \mathbb{R}$ consider the equation

$$\varphi(x) = \lambda^2 \int_{\Omega} \varphi \left(\frac{1}{\lambda} x + \xi(\omega) \right) \mu(d\omega)$$

with $\mu(\Omega) = 1$. According to Lemma 2.1 the zero function is the only its Lipschitzian solution $\varphi: \mathbb{R} \rightarrow \mathbb{R}$. Note however that if

$$\int_{\Omega} \xi(\omega) \mu(d\omega) = 0 \quad \text{and} \quad \int_{\Omega} \xi(\omega)^2 \mu(d\omega) < \infty,$$

then this equation solves also the function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\varphi(x) = x^2 + \frac{\lambda^2}{1 - \lambda^2} \int_{\Omega} \xi(\omega)^2 \mu(d\omega).$$

Example 3.2. Given $\lambda \in (0, 1)$ consider the equation

$$\varphi(x) = 2\varphi(\lambda\sqrt{x} + 1 - \lambda) + \log \frac{x}{(\lambda\sqrt{x} + 1 - \lambda)^2}.$$

According to Lemma 2.1 (in this case $f(x, \omega) = \lambda\sqrt{x} + 1 - \lambda$, $g(\omega) = 2$ and $F(x) = \log \frac{x}{(\lambda\sqrt{x} + 1 - \lambda)^2}$ for all $x \in [1, \infty)$ and $\omega \in \Omega$, $\mu(\Omega) = 1$) the logarithmic function restricted to $[1, \infty)$ is the only Lipschitzian solution $\varphi: [1, \infty) \rightarrow \mathbb{R}$ to this equation, and it is unbounded in spite of the fact that F is bounded.

Example 3.3. To see that assumptions (H_1) – (H_3) do not guarantee the existence of a *continuous* solution $\varphi: X \rightarrow Y$ to equation (1), given $\alpha \in (-1, 1)$, a bounded and \mathcal{A} -measurable $\xi: \Omega \rightarrow \mathbb{R}$, and a Lipschitzian $F: \mathbb{R} \rightarrow [0, \infty)$ such that $F^{-1}(\{0\})$ is a singleton, consider the equation

$$(15) \quad \varphi(x) = \int_{\Omega} \varphi(\alpha x + \xi(\omega)) \mu(d\omega) + F(x)$$

with $\mu(\Omega) = 1$. Assume a continuous $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ solves it. We shall see that then ξ is a.e. constant. To this end fix an $M \in (0, \infty)$ such that $|\xi(\omega)| \leq M$ for every $\omega \in \Omega$, and a real number $a \geq \frac{M}{1-|\alpha|}$ such that $F^{-1}(\{0\}) \subset [-a, a]$. Then

$$|\alpha x + \xi(\omega)| \leq a \quad \text{for all } x \in [-a, a] \text{ and } \omega \in \Omega$$

and so $\varphi|_{[-a, a]}$ is a continuous, hence also bounded, solution of (15). According to [1, Corollary 4.1(ii)] and Example 4.1] it is possible only if ξ is a.e. constant.

4. CONTINUOUS DEPENDENCE

Given a normed space $(Y, \|\cdot\|)$ consider now the linear space $Lip(X, Y)$ of all Lipschitzian functions mapping X into Y , and its linear subspace $BL(X, Y)$ of all Lipschitzian and bounded functions mapping X into Y . Fix $x_0 \in X$ and define $\|\cdot\|_{Lip}: Lip(X, Y) \rightarrow [0, \infty)$ by

$$\|u\|_{Lip} = \|u(x_0)\| + \|u\|_L,$$

where $\|u\|_L$ stands for the smallest Lipschitz constant for u . Clearly $\|\cdot\|_{Lip}$ is a norm in $Lip(X, Y)$. It depends on the fixed point x_0 , but for different points such norms are equivalent. It is well known that if $(Y, \|\cdot\|)$ is Banach, then so is $(Lip(X, Y), \|\cdot\|_{Lip})$. In the linear space $BL(X, Y)$ we consider the norm $\|\cdot\|_{BL}$ given by

$$\|u\|_{BL} = \sup \{ \|u(x)\| : x \in X \} + \|u\|_L.$$

It is also well known that if $(Y, \|\cdot\|)$ is Banach, then so is $(BL(X, Y), \|\cdot\|_{BL})$.

Assume (H_1) and (H_2) , $\gamma \neq 1$, and let Y be a separable Banach space over \mathbb{K} .

According to Theorem 2.3 for every $F \in Lip(X, Y)$ the formula

$$(16) \quad \varphi^F(x) = \frac{1}{1-\gamma} \left(\sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + F(x) \right)$$

for every $x \in X$, defines the only Lipschitzian solution φ^F of equation (1),

$$(17) \quad \|\varphi^F\|_L \leq \frac{1+|\gamma|}{|1-\gamma|(1-\lambda)} \|F\|_L$$

and

$$(18) \quad \|\varphi^F(x)\| \leq \frac{1}{|1-\gamma|} \left(\frac{\|F\|_L}{1-\lambda} \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) + \|F(x)\| \right)$$

for every $x \in X$. Putting

$$(19) \quad c_0 = \frac{1}{1-\lambda} \left(\int_{\Omega} |g(\omega)| \rho(f(x_0, \omega), x_0) \mu(d\omega) + 1 + |\gamma| \right), \quad c = \max\{1, c_0\},$$

and applying (17) and (18) we see that if $F \in Lip(X, Y)$, then

$$\begin{aligned} \|\varphi^F\|_{Lip} &= \|\varphi^F(x_0)\| + \|\varphi^F\|_L \leq \frac{1}{|1-\gamma|} (c_0 \|F\|_L + \|F(x_0)\|) \\ &\leq \frac{c}{|1-\gamma|} \|F\|_{Lip}. \end{aligned}$$

Moreover, if d_0 defined by

$$(20) \quad d_0 = \sup \left\{ \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) : x \in X \right\}$$

is finite, then putting

$$(21) \quad d = \max \left\{ 1, \frac{d_0 + 1 + |\gamma|}{1-\lambda} \right\}$$

and applying (18) and (17) again we see also that if $F \in BL(X, Y)$, then $\varphi^F \in BL(X, Y)$ as well and

$$\|\varphi^F\|_{BL} \leq \frac{1}{|1-\gamma|} \left(\frac{d_0 + 1 + |\gamma|}{1-\lambda} \|F\|_L + \sup \{ \|F(x)\| : x \in X \} \right) \leq \frac{d}{|1-\gamma|} \|F\|_{BL}.$$

Theorem 4.1. *Assume (H_1) , (H_2) and let γ defined by (4) be different from 1. If Y is a separable Banach space over \mathbb{K} then:*

(i) *for any $F \in Lip(X, Y)$ the function $\varphi^F : X \rightarrow Y$ defined by (16) and (6) is the only Lipschitzian solution of (1), the operator*

$$(22) \quad F \mapsto \varphi^F, \quad F \in Lip(X, Y),$$

is a linear homeomorphism of $(Lip(X, Y), \|\cdot\|_{Lip})$ onto itself and

$$\|\varphi^F\|_{Lip} \leq \frac{c}{|1-\gamma|} \|F\|_{Lip} \quad \text{for every } F \in Lip(X, Y)$$

with c given by (19);

(ii) if additionally d_0 defined by (20) is finite, then the restriction of the operator (22) to $BL(X, Y)$ is a linear homeomorphism of $(BL(X, Y), \|\cdot\|_{BL})$ onto itself and

$$\|\varphi^F\|_{BL} \leq \frac{d}{|1-\gamma|} \|F\|_{BL} \quad \text{for every } F \in BL(X, Y)$$

with d given by (21).

Proof. By the above considerations and the Banach inverse mapping theorem it remains to show that operator (22) is one-to-one, maps $Lip(X, Y)$ onto $Lip(X, Y)$ and $BL(X, Y)$ onto $BL(X, Y)$.

The first property follows from the fact that for any $F \in Lip(X, Y)$ the function φ^F is a solution of (1): if $\varphi^F = 0$, then $F = 0$. To get the next two observe that if $\psi \in Lip(X, Y)$, then by Lemma 2.2 the function $F: X \rightarrow Y$ given by

$$F(x) = \psi(x) - \int_{\Omega} g(\omega) \psi(f(x, \omega)) \mu(d\omega)$$

belongs to $Lip(X, Y)$, if ψ is also bounded, then so is F , and, since both ψ and φ^F solve (1), $\psi = \varphi^F$ by Lemma 2.1. \square

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